**Static Dielectric**

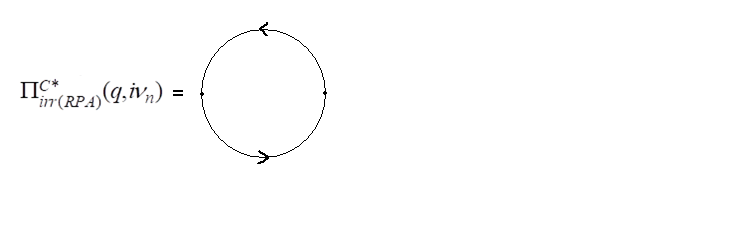
So I guess we’re doing the electric susceptibility stuff. We’ll presume we’re in a time-independent situation, and so everything has come to equilibrium. Or possibly we could allow time-dependence, as long as it is much slower than the thermal relaxation time. In this case, time would just be a spectator variable. The formalism below is found in the EM folder too.

**RPA approximation to ε**

So recall we have:



where V(q) = e2υ(q) = 4πe2/q2. So our estimate of ε(q,0) boils down to an estimate of Πirr(q,0). In the RPA we use the first order approximation to ΠirrR. This approximation works best for high densities. That makes sense because metals are essentially ‘free’ at high densities (because at these densities KE >> PE), and this is a non-interacting result for Πirr (literally, as there is no interaction lines in Πirr above). But this won’t cover most metals. Although, also turns out that our result will be exact in the limit that q → 0, regardless of e-e interactions.



Can quickly evaluate this, like we did in the Collective excitations file. So we have to lowest order:



Since G0C\*(k,iωn) has just one isolated pole, it is straightforward to use the contour integration method to evaluate.



And then,



which is the Lindhardt function from before. And we’re interested in the iνn = 0 case.



Can look at the Free Day folder for more on the properties of this function. We found that at T = 0,



and looks like this:

Chart, line chart

Description automatically generated

There is a logarithmic singularity at k = 2kF (x = 1), as one can verify. Interesting. It follows that:



and,



**Thomas-Fermi Approximation**

Let’s go to the small q limit. Then we have:



We’ll actually just take the very first term, 1. Then we’d have:



And so we have:



And we’ll recognize this as the Thomas-Fermi approximation to the susceptibility (see Free Day/Electrons). Can/will also write as:



And it follows that:



which is of course,



Observe that in the long-wavelength limit, ε(q) → ∞ as we’d expect (very good at screening long wavelength fields). And then last,



which comes to:



Let’s consider the induced charge when we place an free charge within the medium.



Let’s say φf(r) = e/r [Gaussian units, again]. Then φf(q) = 4πe/q2. Filling that in we get:



Taking the inverse FT, we find:



and the induced potential would be given by the Laplacian of this quantity. And let’s do φ in terms of φf. We have:



Likewise taking the inverse FT we’d have:



We can extend the integral to whole real line since it’s even,



Now have to use residue theorem. If r is positive then we close contour up, and if negative then down, and we get:



So it’s just the bare potential, with an exponential decay factor attached. Might be nice to know what the decay length is, for real. So we should convert that exponent to SI units. We are using Natural + Gaussian units (see Units file) and so we’ve set μ0 = 4π and ε0 = 1/4π, and ℏ = 1. So our exponent should really look like:



where p, q, and r are unknown as of yet. Well the exponent must work out unitless in SI (or any unit system). In SI the units of those guys are:



And so we demand:



So we need 2p = r, which makes things out to:



Middle one requires q = p + ½, so,



This requires p = -1/2, and so q = 0, and so r = -1. So that exponent should be:



Kind of a laborious process to put this back into the correct units, but I guess it’s better than lugging ε0’s, μ0’s, and ℏ’s around. So the characteristic length scale over which the potential decays is:



Might note that:



So we can say,



Now for Cu, kF ~ 1/0.7nm, and this works out to about ℓTF ~ 0.52nm, basically the Bohr radius. So that’s a really tight screening radius. Can see why it’s often said, in introductory physics, that metals will completely screen out external electric fields. So while that’s not *completely* true, it’s pretty close. FWIW, we can now write out χirr in SI units too (see Units file),



So there.

**Friedel Oscillations**

Now normally, if we’re interested in the large r behavior ρ (or any function of r), then it suffices to use the small q behavior of ρ. And thus it usually suffices to use the small q approximation to χ(q). But this presumes χ(q) is analytic (apparently), whereas we know it is *not* near q = 2kF. And if we do this calculation more carefully, using the full F(x),



instead of F(x) ≈ 1, then we find the induced charge decays in an oscillatory algebraic fashion, not exponentially.



The oscillations are called Friedel oscillations. The total potential also has an oscillating tail as well. It would be given by:



where,



Not going to do this, but result, for large r, is:



Well, we’ll notice that this also indicates that the potential’s amplitude decay doesn’t happen on an atomic length scale. But the rapid oscillations, which *do* happen over an atomic scale, would seem to compensate for this, and make the potential practically/averagelly zero outside of a lattice spacing ~ 1/kF. Apparently, at finite T, the sharp cutoff at k = 2kF is smeared and the oscillations are weakened by a multiplicative factor exp(-2πkBT|r|/vF), where vF is Fermi velocity.